§ Rational Root Theorem

Rational Root Thm: Let $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$,
if $\frac{p}{q}$ is a root, then p is a factor
of a_n & q is a factor of a_0.

Morgan's version: given a polynomial of $Ax^n + bx^{n-1} + \ldots + C = 0$
if $\frac{p}{q}$ is a root of the poly,
then p is a factor of C & q is a factor of A.

meaning the possible roots of the poly would be:

- factors $^{(p)}$ of C
- factors $^{(q)}$ of A
- ratios of $\frac{p}{q}$

Example: $\underset{A}{6}x^3 + 11x^2 - 3x - \underset{C}{2} = 0$

- possible factors for $p(C)$: $\pm 1, \pm 2$
- possible factors for $q(A)$: $\pm 1, \pm 2, \pm 3, \pm 6$
- possible factors for $\frac{p}{q}$: $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm 2, \pm \frac{2}{2}, \pm \frac{2}{3}, \pm$

* these are the possible factors

* The rational root theorem tells you the possible roots

Rational Root Thm Cont...

ex: Given $2x^3 + 3x^2 - 8x + 3 = 0$

possible p: $\pm 1, \pm 3$

possible q: $\pm 1, \pm 2$

$$\boxed{\text{possible } \frac{p}{q}: \pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2}}$$

these are the possible roots

- to determine which possible roots actually work, use synthetic division.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & +3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0^{*} \end{array}$$

* this will be a guess & check routine. there are 8 possible roots, but we know the equation only has 3 roots.

* since 1 gave us 0 we know 1 is a root

take $2x^2 + 5x - 3$

from here factor or quadratic ~~equation~~ formula

Factor

$2x^2 + 5x - 3$

$(2x - 1)(x + 3)$

$x = \frac{1}{2}, -3$

Quadratic Formula

$$= \frac{-5 \pm \sqrt{5^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{-5 \pm \sqrt{49}}{4}$$

$$= \frac{-5 \pm 7}{4} = -3, \frac{1}{2}$$

$\therefore$ roots are $1, \frac{1}{2}, -3$

Rational Root Theorem

Example: $x^3-4x^2+x+2=0$

p: ±1, ±2

q: ±1

$\boxed{\frac{p}{q}: \pm1, \pm2}$ possible roots

$$\begin{array}{r|rrrrl} 1 & 1 & -4 & 1 & 2 & \\ & & 1 & -3 & -2 & \\ \hline & 1 & -3 & -2 & 0 & \text{root} \end{array}$$

$x^2-3x-2=0$

~~x~~ $\dfrac{-(-3)\pm\sqrt{(-3)^2-4(1)(-2)}}{2(1)}$

$x=\dfrac{3\pm\sqrt{17}}{2}$

roots: 1, $\dfrac{3\pm\sqrt{17}}{2}$

Example: $6x^4-7x^3-4x^2+7x-2$

p: ±1, ±2

q: ±1, ±2, ±3, ±6

$\boxed{\frac{p}{q}: \pm1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm2, \pm\frac{2}{3},}$ possible roots

Find roots

$$\begin{array}{r|rrrrrl} \frac{1}{2} & 6 & -7 & -4 & 7 & -2 & \\ & & 3 & -2 & -3 & 2 & \\ \hline 1 & 6 & -4 & -6 & 4 & 0 & \text{root} \\ & & 6 & 2 & -4 & & \\ \hline -1 & 6 & 2 & -4 & 0 & & \text{root} \\ & & -6 & 4 & & & \\ \hline & 6 & -4 & 0 & & & \text{root} \end{array}$$

* not left with quadratic use synthetic again

$6x-4=0$

$6x=4$

$x=\frac{2}{3}$ root

Actual roots: $\frac{1}{2}$, 1, −1, $\frac{2}{3}$

Descartes Rule of Signs

Descartes rule of signs - helps determine the

- possible number of positive real zeros,
- possible number of negative real zeros,
- possible number of imaginary roots.

- To determine possible positive real zeros, count the # of sign changes in $f(x)$.

<u>Given:</u> $2x^5 + 3x^4 - 6x^3 + 6x^2 - 8x + 3$

$$+ \quad \underbrace{+ \quad - \quad + \quad - \quad +}_{1 \quad 2 \quad 3 \quad 4}$$

signs change 4 times

possible positive reals: 4 or 2 or 0

- To determine possible negative real zeros count the # of sign changes in $f(\underline{\underline{-x}})$

$$2(-x)^5 + 3(-x)^4 - 6(-x)^3 + 6(-x)^2 - 8(-x) + 3$$

$$\underbrace{- \quad +}_{1} \quad + \quad + \quad + \quad +$$

signs change once

possible negative reals: 1

- To determine # of possible imaginary, all roots have to add to equal possible # complex roots

* degree is 5

∴ 5 complex roots

\+ real: 4 or 2 or 0

\- real: 1

imag: 0 or 2 or 4

* imag always even set

Descartes cont. . .

Example: $f(x) = x^3 - x^2 - 14x - 6$ ⇒ 3 complex roots

poss + real: $+ \quad - \quad - \quad -$ 1 sign change

poss − real: $(-x)^3 - (-x)^2 - 14(-x) - 6$

$- \quad - \quad + \quad -$ 2 sign change

+ real: 1

− real: 2 or 0

imag: 0 or 2

$$\begin{array}{r} 1 \\ +2 \\ \hline 3 \end{array}$$ so no imag

$$\begin{array}{r} 1 \\ +0 \\ \hline 1 \end{array}$$ ∴ 2 roots left, so must be imag.

Example: $f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$ ⇒ 4 complex roots

poss + real: $+ \; \underbrace{+ \; -}_{1} \; \underbrace{- \; +}_{2}$: 2 or 0

poss − real: $(-x)^4 + 2(-x)^3 - 9(-x)^2 - 2(-x) + 8$

$\underbrace{+ \; -}_{1} \; \underbrace{- \; +}_{2} \; +$: 2 or 0

+ real: 2 or 0

− real: 2 or 0

imag: 0 or 2

Example: Given $f(x) = x^4 + 2x^3 - 3x^2 - 8x - 4$

Determine: ⓐ # of complex roots

ⓑ # of possible positive, negative, & imaginary roots

ⓒ list the possible rational zeros

ⓓ find the zeros.

ⓐ 4 complex roots

ⓑ + real: + + − − − : 1

− real: + − + − − : 3 or 1

imag: : 0 or 2

ⓒ $p: \pm 1, \pm 2, \pm 4$

$q: \pm 1$

$$\boxed{\frac{p}{q}: \pm 1, \pm 2, \pm 4}$$

ⓓ

$$\begin{array}{r|rrrrr}
-2 & 1 & 2 & -3 & -8 & -4 \\
 & & -2 & 0 & 6 & +4 \\
\hline
-1 & 1 & 0 & -3 & -2 & \| \; 0 \text{ root} \\
 & & -1 & 1 & 2 & \\
\hline
 & 1 & -1 & -2 & \| \; 0 \text{ root} &
\end{array}$$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

$$\boxed{\text{roots: } -1, -1, -2, 2}$$